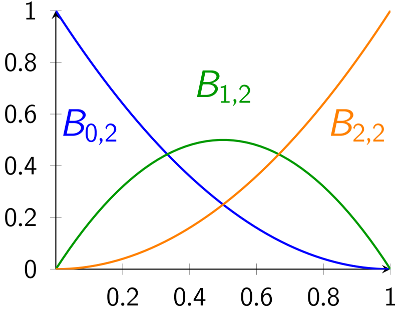
Bestimmung von Polynomnullstellen mittels Clipping

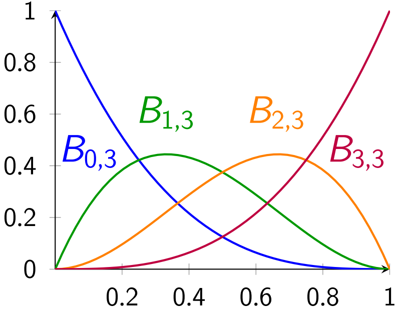
Timo Bingmann

17. März 2012

Übersicht

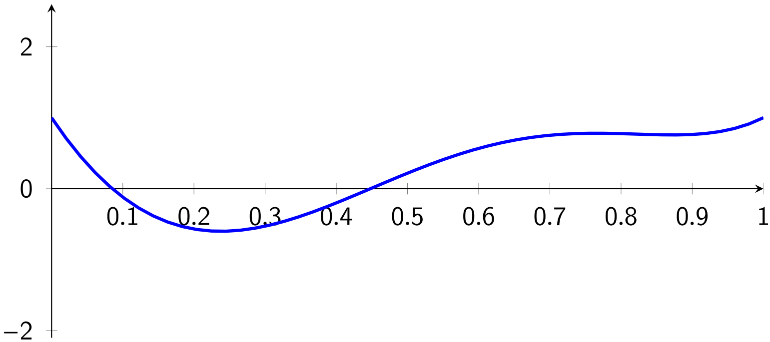
Bernstein-Grundpolynome





Bézier-Clipping

$$\begin{aligned} p &= 25X^5 - 35X^4 - 15X^3 + 40X^2 - 15X + 1 \\ &= 1B_{0,5}(X) - 2B_{1,5}(X) - 1B_{2,5}(X) \\ &\qquad + 2.5B_{3,5}(X) + 0B_{4,5}(X) + 1B_{5,5}(X) \end{aligned}$$



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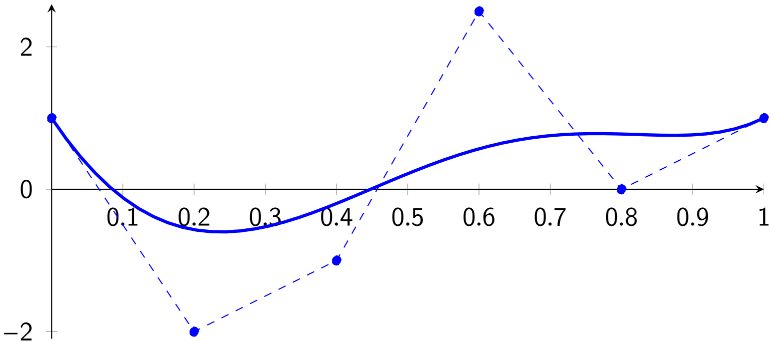

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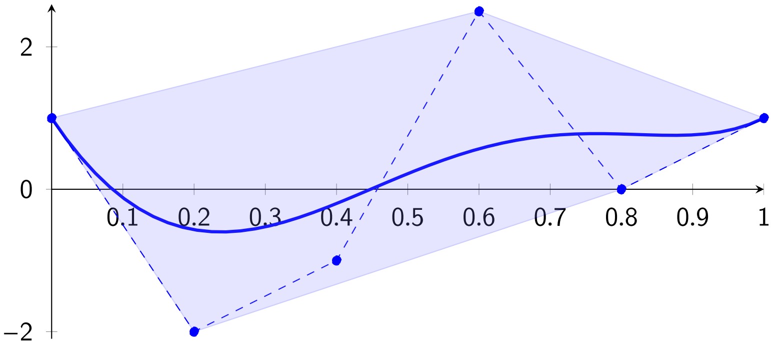

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\end{aligned}
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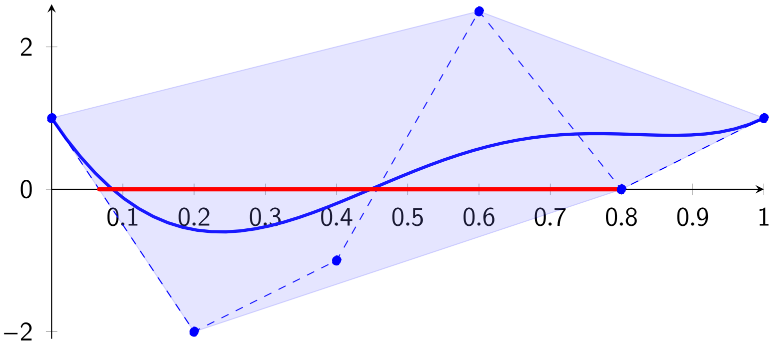

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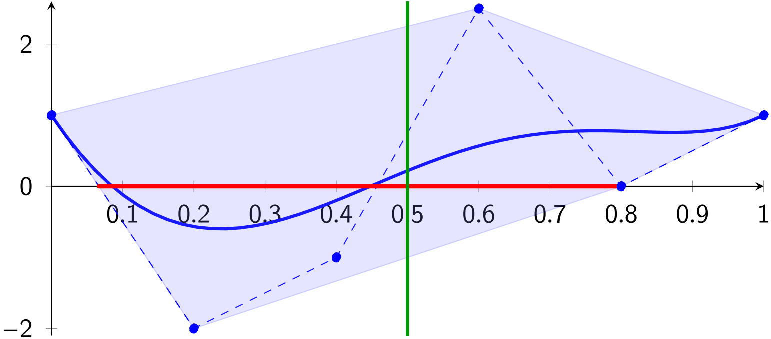

Algorithmus von de Casteljau für 0.5

1

-2

-1

2.5

0

1

Algorithmus von de Casteljau für 0.5

1	
	−0.5
−2	
	−1.5
−1	
	0.75
2.5	
	1.25
0	
	0.5
1	

Algorithmus von de Casteljau für 0.5

1		
	−0.5	
−2		−1
	−1.5	
−1		−0.375
	0.75	
2.5		1
	1.25	
0		0.875
	0.5	
1		

Algorithmus von de Casteljau für 0.5

1					
	−0.5				
−2		−1			
	−1.5		−0.6875		
−1		−0.375		−0.1875	
	0.75		0.3125		0.21875
2.5		1		0.625	
	1.25		0.9375		
0		0.875			
	0.5				
1					

Algorithmus von de Casteljau für 0.5

1					
	-0.5				
-2		-1			
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1					

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	0.5				
1					

Intervallhalbierung mit de Casteljau

$$p = 25X^5 - 35X^4 - 15X^3 + 40X^2 - 15X + 1$$
$$= 1B_{0,5}(X) - 2B_{1,5}(X) - 1B_{2,5}(X)$$
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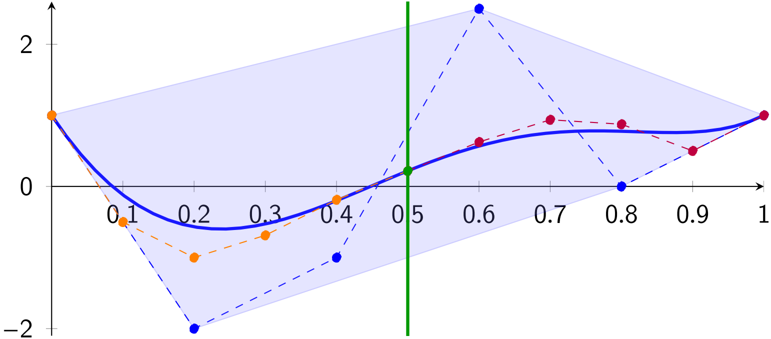
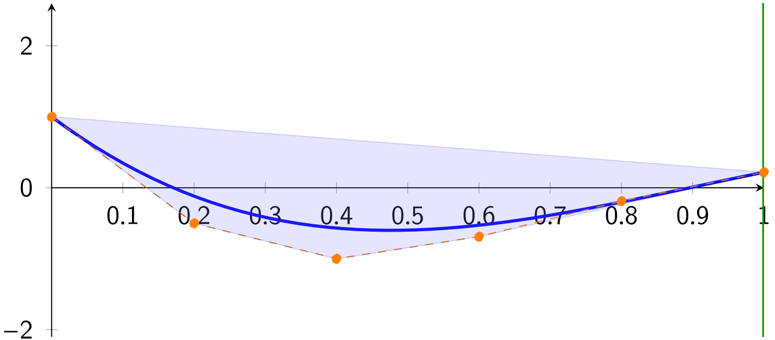


Intervallhalbierung mit de Casteljau

$$p = 1B_{0,5}(X) - 0.5B_{1,5}(X) - 1B_{2,5}(X) - 0.6875B_{3,5}(X) \\ - 0.1875B_{4,5}(X) + 0.21875B_{5,5}(X)$$



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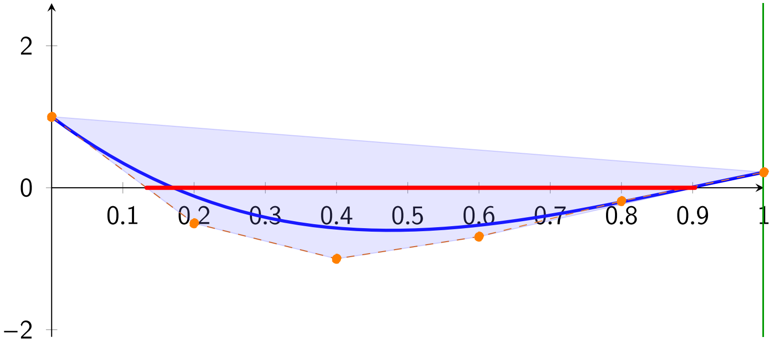

Intervallhalbierung mit de Casteljau

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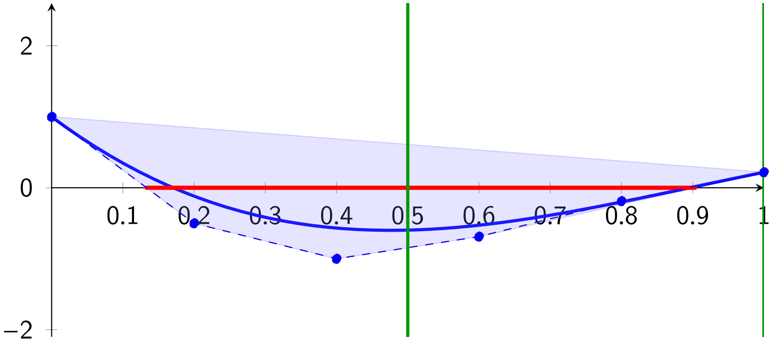

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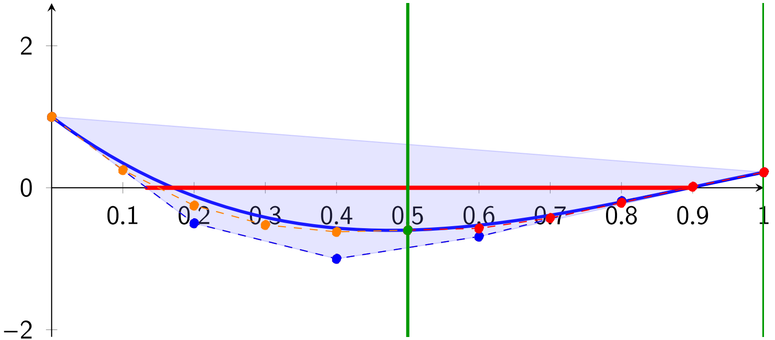
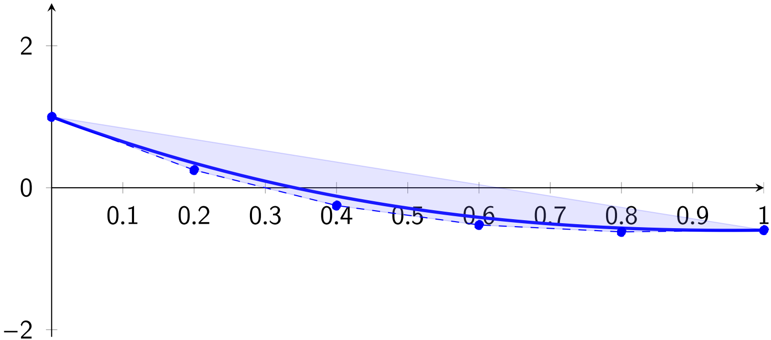


Intervallhalbierung mit de Casteljau

$$p = 1B_{0,5}(X) + 0.25B_{1,5}(X) - 0.25B_{2,5}(X) - 0.523438B_{3,5}(X) - 0.621094B_{4,5}(X) - 0.59668B_{5,5}(X)$$



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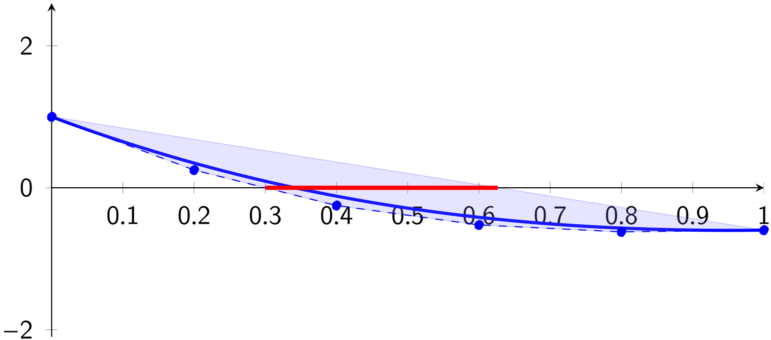

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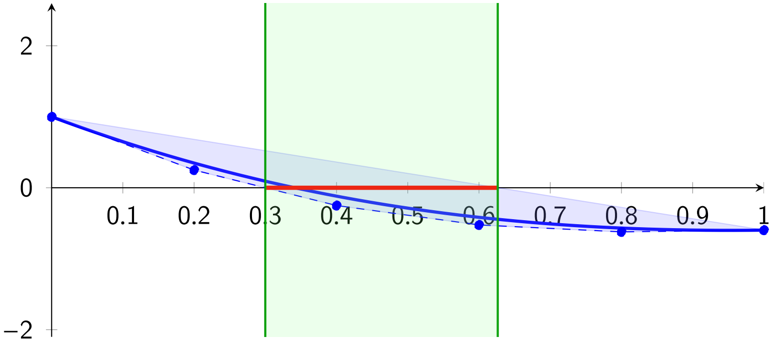


Übersicht

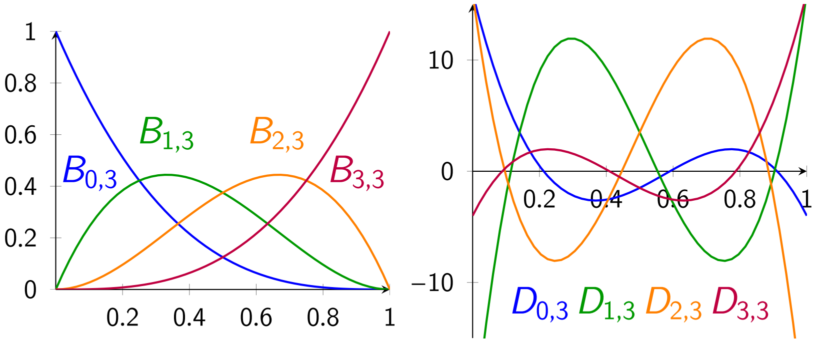
Reziproke Basis $D_{i,n}(t)$

$$D_{i,n}(t) = \sum_{j=0}^{n} c_{i,j} B_{j,n}(t)$$

$$c_{i,j} = \frac{(-1)^{i+j}}{\binom{n}{i}\binom{n}{j}} \cdot \sum_{k=0}^{\min(i,j)} (2k+1)\, U(i)\, U(j)$$

$$\text{mit } U(r) = \binom{n+k+1}{n-r}\binom{n-k}{n-r}$$

Reziproke Basis $D_{i,n}(t)$



Berechnung von $M^{(N,n)} := (\beta_{i,j}^{(N,n)})$

$$\begin{aligned}
\beta_{i,j}^{(N,n)} &= \langle B_{i,N}, D_{j,n} \rangle \\
&= \left\langle B_{i,N}, \sum_{k=0}^{n} c_{j,k} B_{k,n} \right\rangle \\
&= \sum_{k=0}^{n} c_{j,k} \langle B_{i,N}, B_{k,n} \rangle
\end{aligned}$$

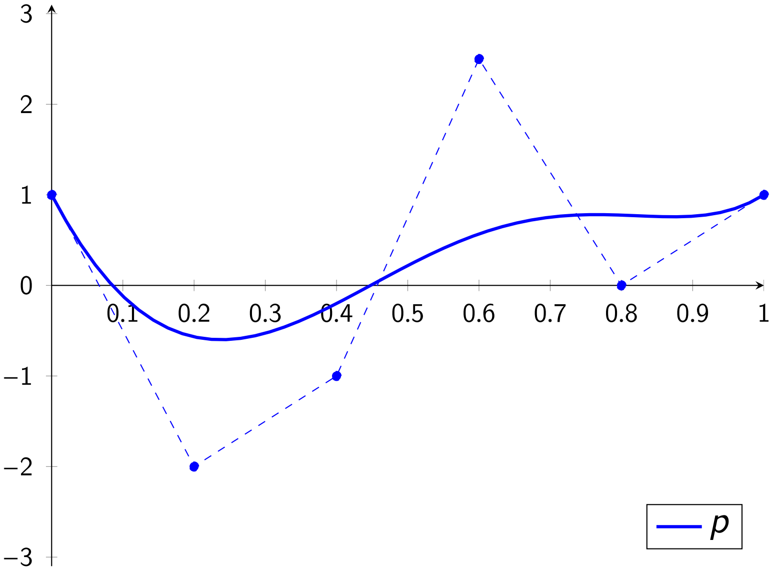
direkt lösbar durch

$$\langle B_{i,m}, B_{j,n} \rangle = \frac{\binom{m}{i}\binom{n}{j}}{(m+n+1)\binom{m+n}{i+j}}$$

Beispiele für $M^{(N,n)}$

$$M^{(5,2)} = \begin{pmatrix} \frac{23}{28} & \frac{-3}{7} & \frac{3}{28} \\ \frac{9}{28} & \frac{2}{7} & \frac{-3}{28} \\ 0 & \frac{9}{14} & \frac{-1}{7} \\ \frac{-1}{7} & \frac{9}{14} & 0 \\ \frac{-3}{28} & \frac{2}{7} & \frac{9}{28} \\ \frac{3}{28} & \frac{-3}{7} & \frac{23}{28} \end{pmatrix} \quad M^{(5,3)} = \begin{pmatrix} \frac{121}{126} & \frac{-3}{7} & \frac{1}{6} & \frac{-2}{63} \\ \frac{8}{63} & \frac{37}{42} & \frac{-3}{7} & \frac{11}{126} \\ \frac{-1}{9} & \frac{16}{21} & \frac{1}{21} & \frac{-2}{63} \\ \frac{-2}{63} & \frac{1}{21} & \frac{16}{21} & \frac{-1}{9} \\ \frac{11}{126} & \frac{-3}{7} & \frac{37}{42} & \frac{8}{63} \\ \frac{-2}{63} & \frac{1}{6} & \frac{-3}{7} & \frac{121}{126} \end{pmatrix}$$

Anwendung der Grad-Reduktionstechnik



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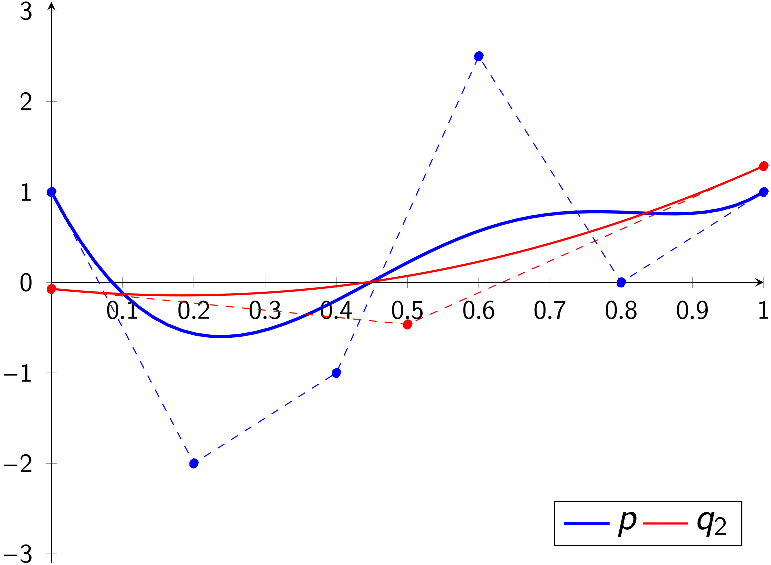

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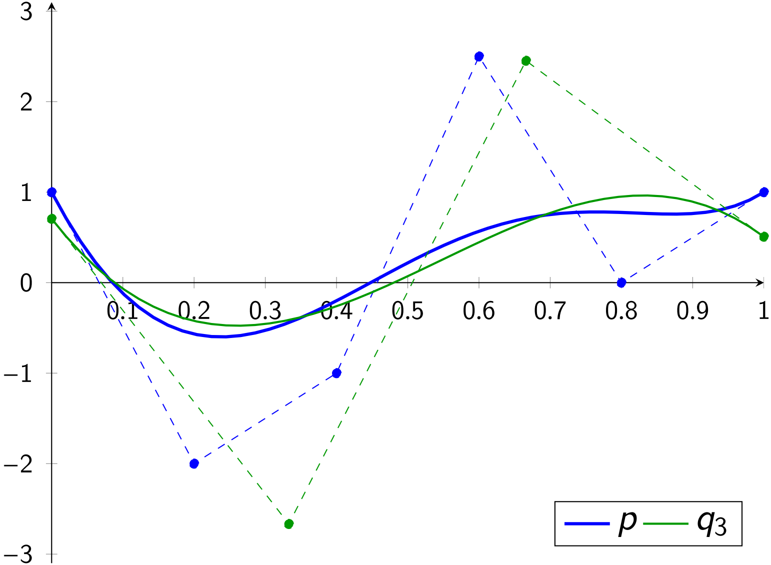

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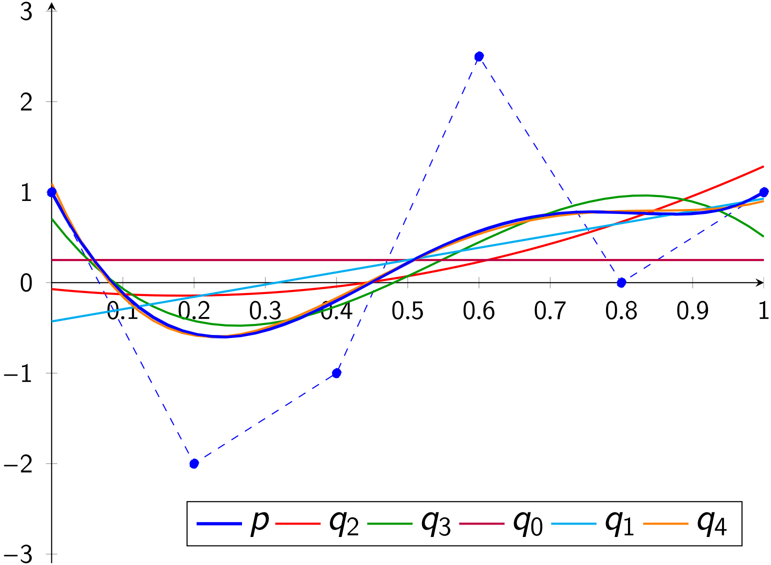
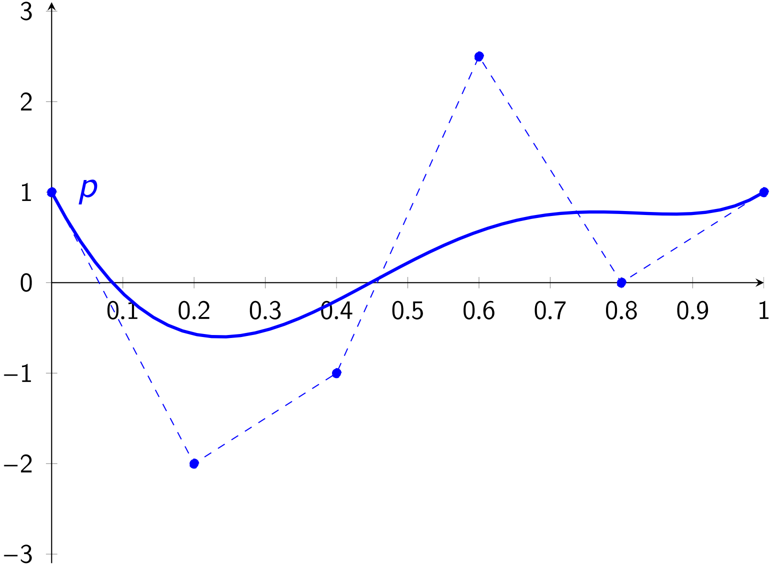


Übersicht

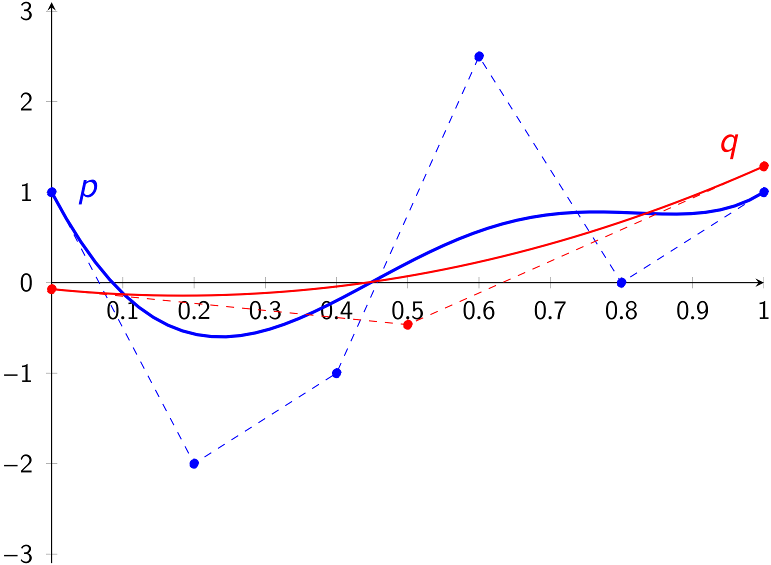
Der QuadClip Algorithmus



Grad-Reduktion mit $M^{(5,2)}$

$$M^{(5,2)} \cdot b = \begin{pmatrix} \frac{23}{28} & \frac{-3}{7} & \frac{3}{28} \\ \frac{9}{28} & \frac{2}{7} & \frac{-3}{28} \\ 0 & \frac{9}{14} & \frac{-1}{7} \\ \frac{-1}{7} & \frac{9}{14} & 0 \\ \frac{-3}{28} & \frac{2}{7} & \frac{9}{28} \\ \frac{3}{28} & \frac{-3}{7} & \frac{23}{28} \end{pmatrix}^{t} \begin{pmatrix} 1 \\ -2 \\ -1 \\ 2.5 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -0.0714286 \\ -0.464286 \\ 1.28571 \end{pmatrix}$$

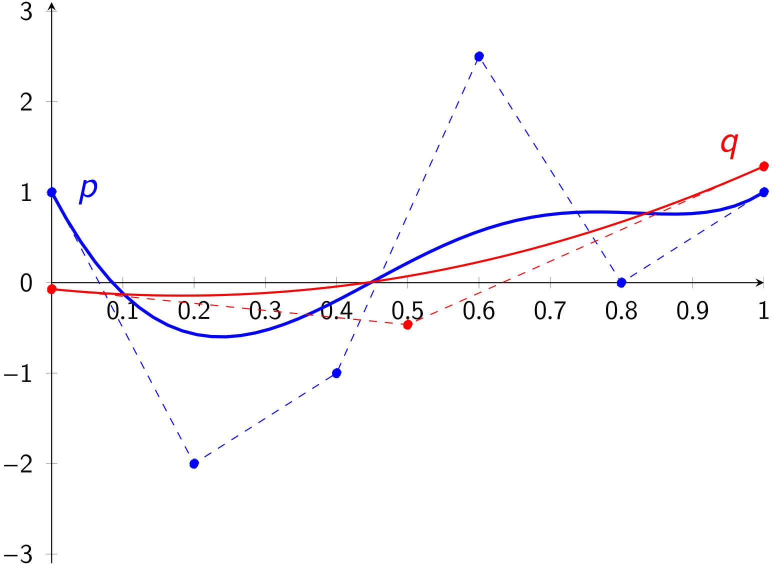
Der QuadClip Algorithmus



Grad-Erhöhung mit $M^{(2,5)}$

$$\begin{pmatrix} 1 & \frac{3}{5} & \frac{3}{10} & \frac{1}{10} & 0 & 0 \\ 0 & \frac{2}{5} & \frac{3}{5} & \frac{3}{5} & \frac{2}{5} & 0 \\ 0 & 0 & \frac{1}{10} & \frac{3}{10} & \frac{3}{5} & 1 \end{pmatrix}^{t} \begin{pmatrix} -0.07142 \\ -0.4642 \\ 1.28571 \end{pmatrix} = \begin{pmatrix} 0.706349 \\ -1.31746 \\ -0.793651 \\ 0.722222 \\ 1.6746 \\ 0.507937 \end{pmatrix}$$

Der QuadClip Algorithmus



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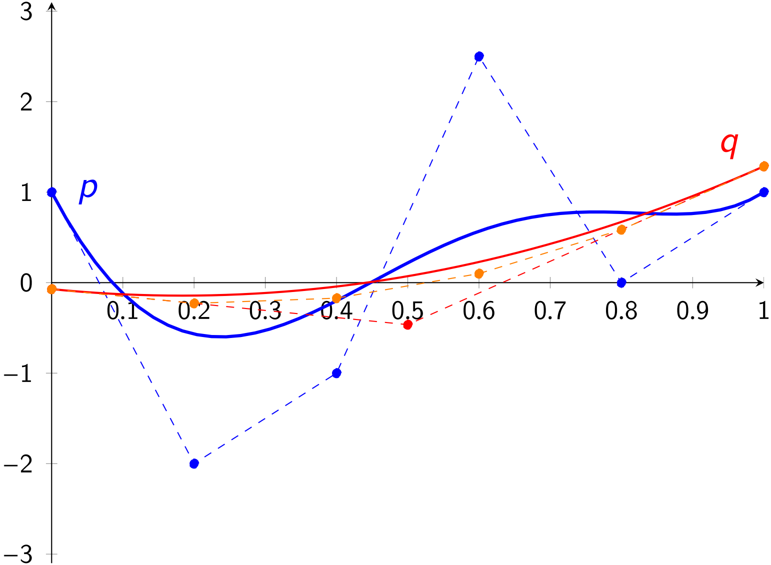

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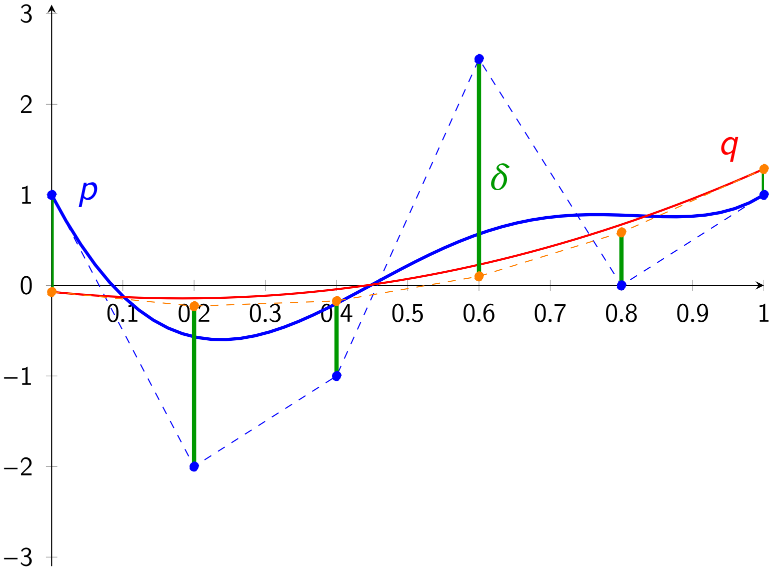

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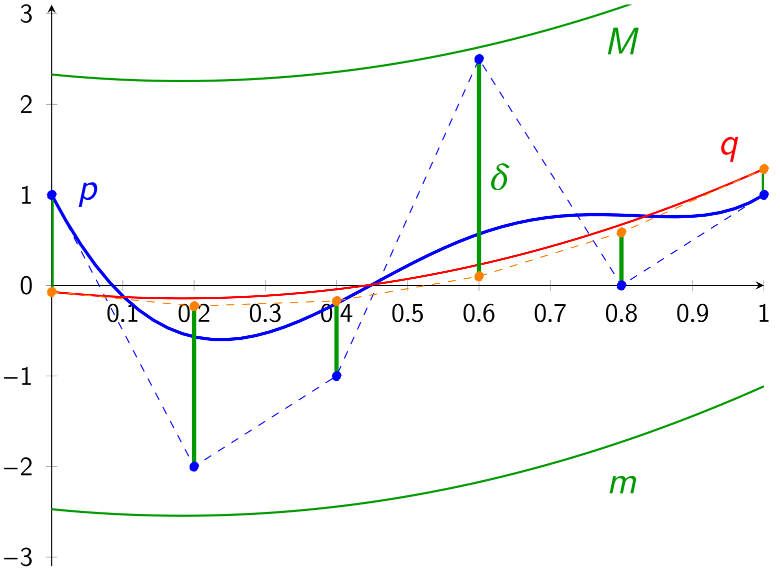

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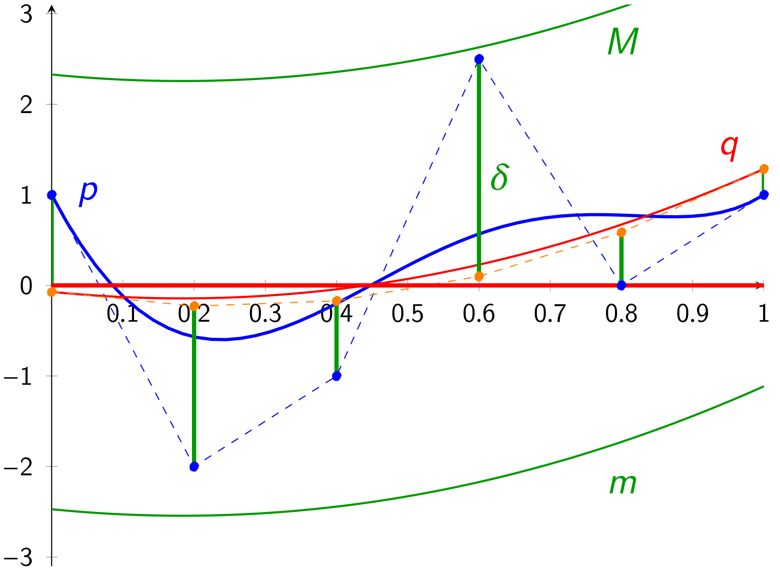

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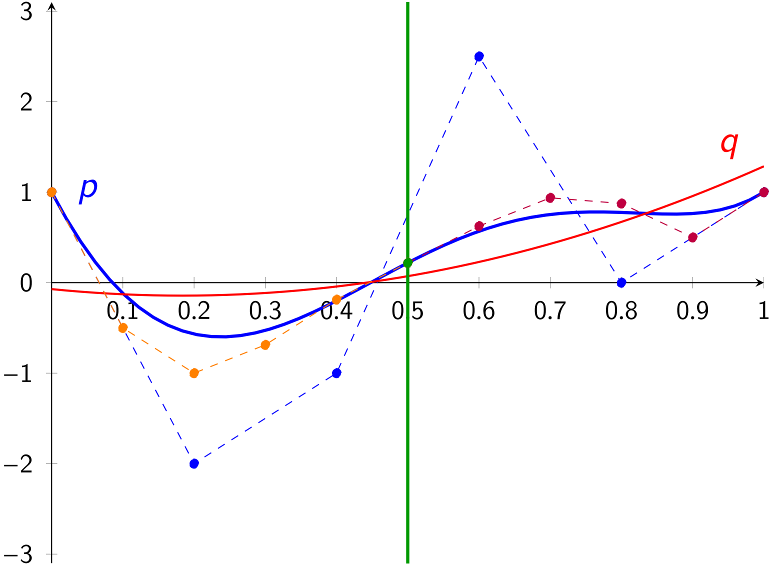

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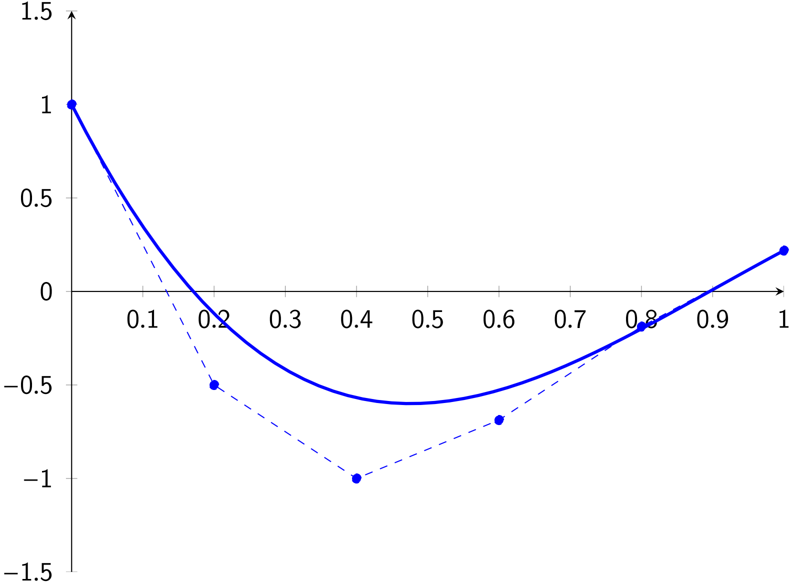

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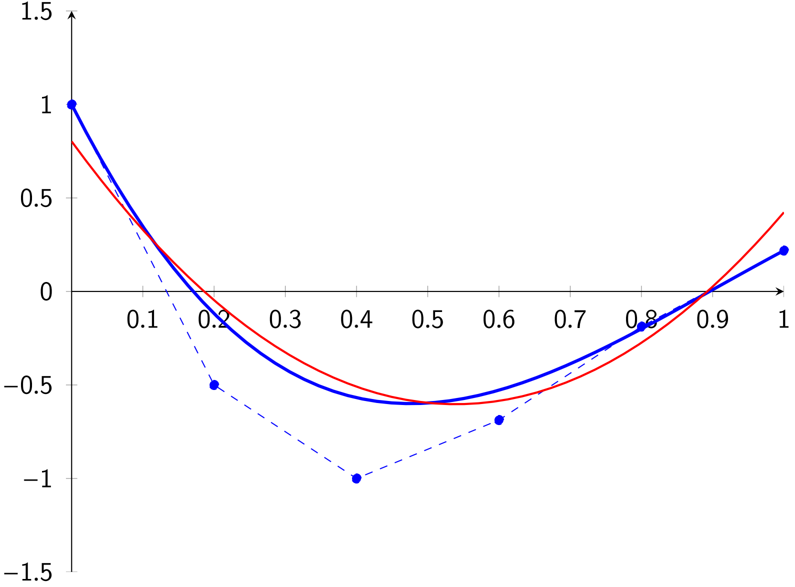

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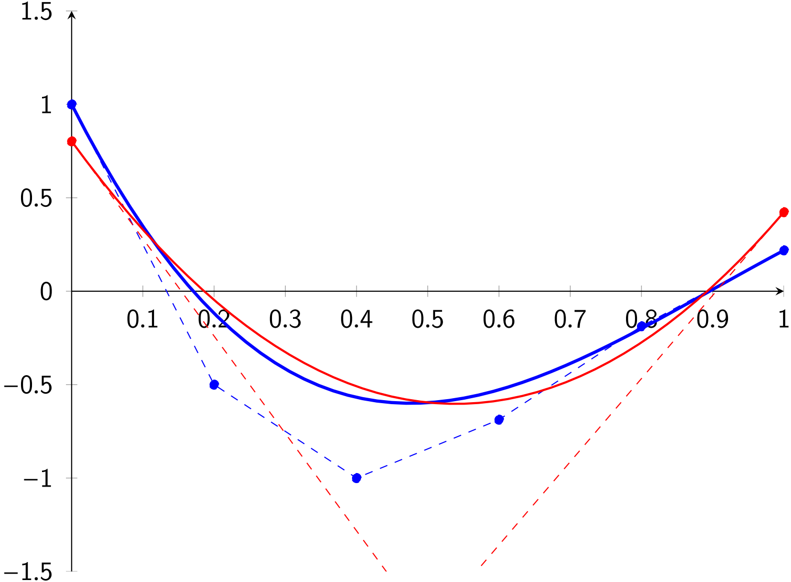

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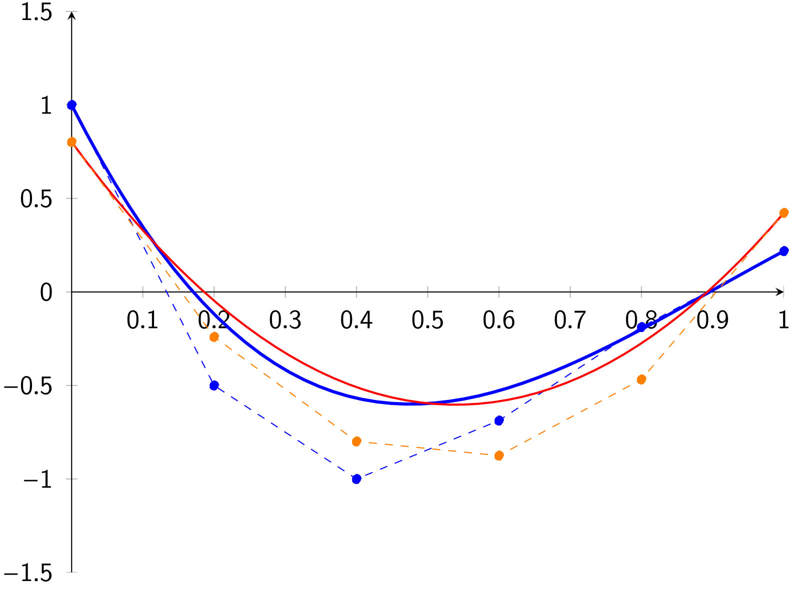

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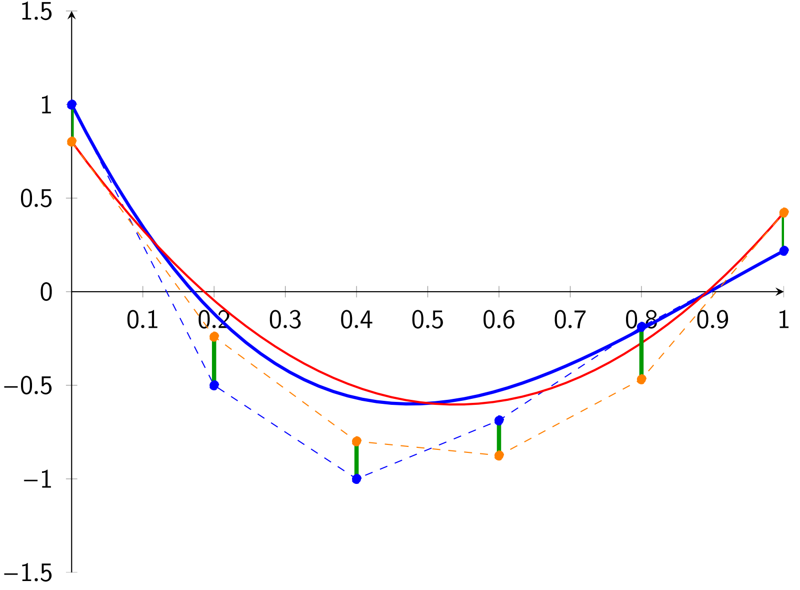

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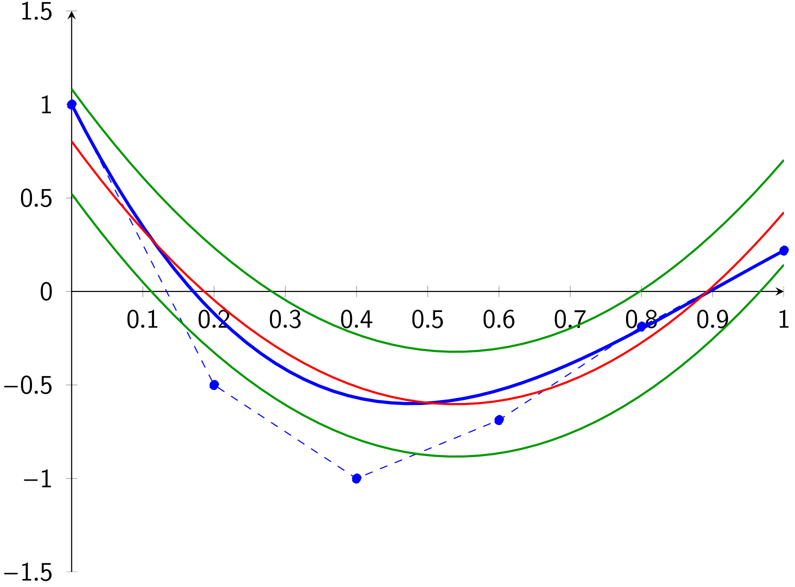

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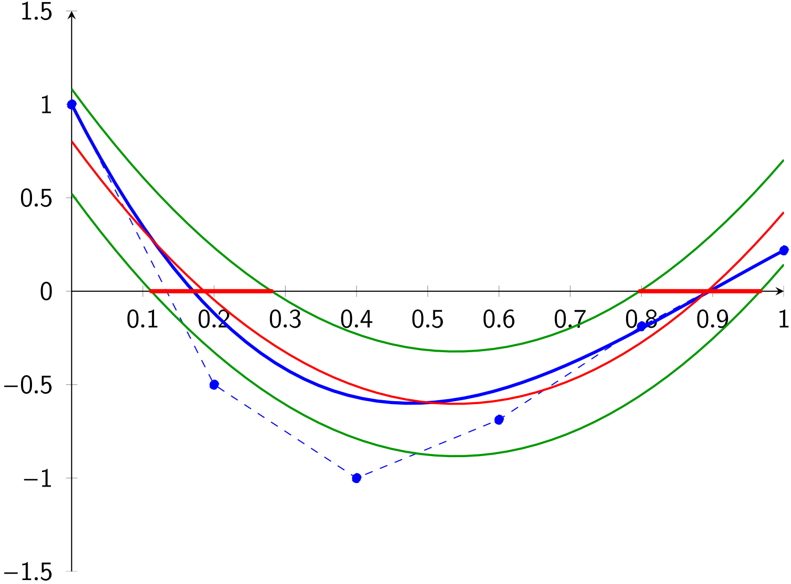

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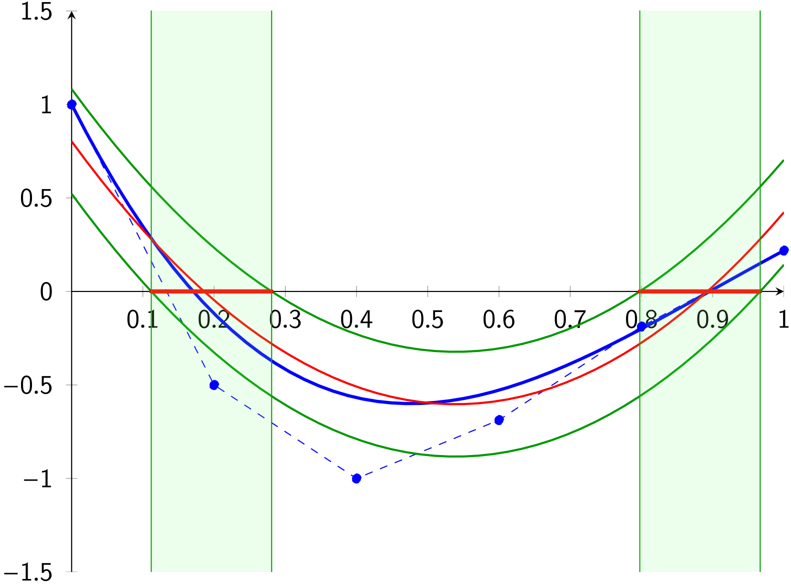

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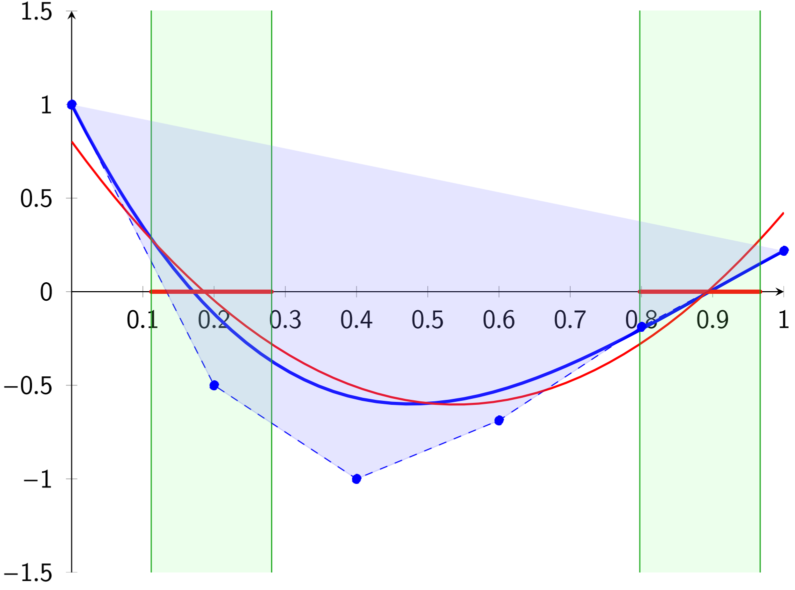

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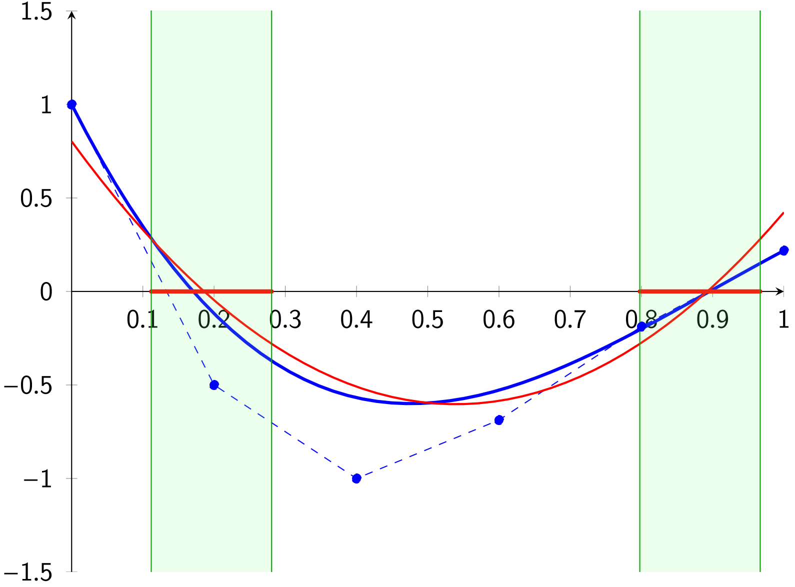
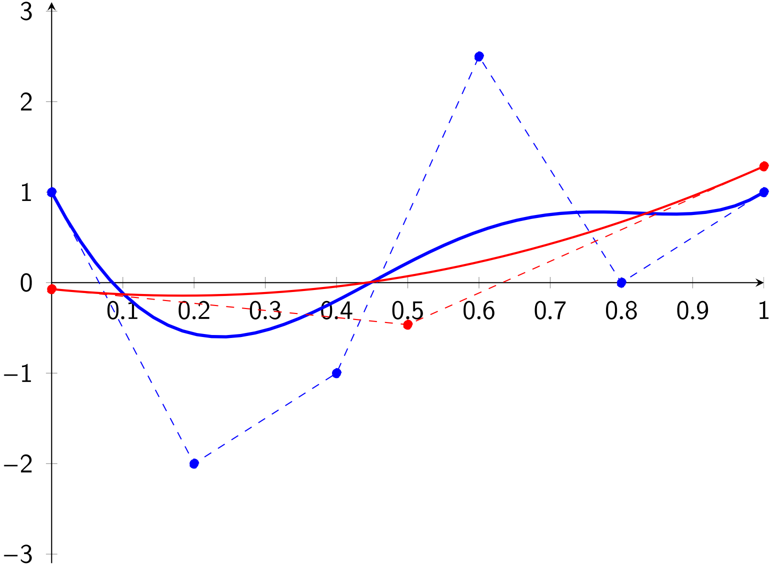


Der CubeClip Algorithmus



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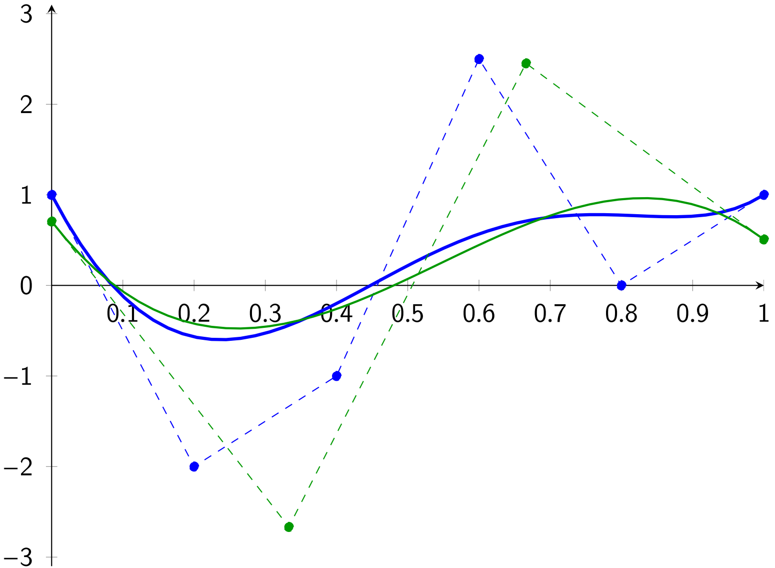

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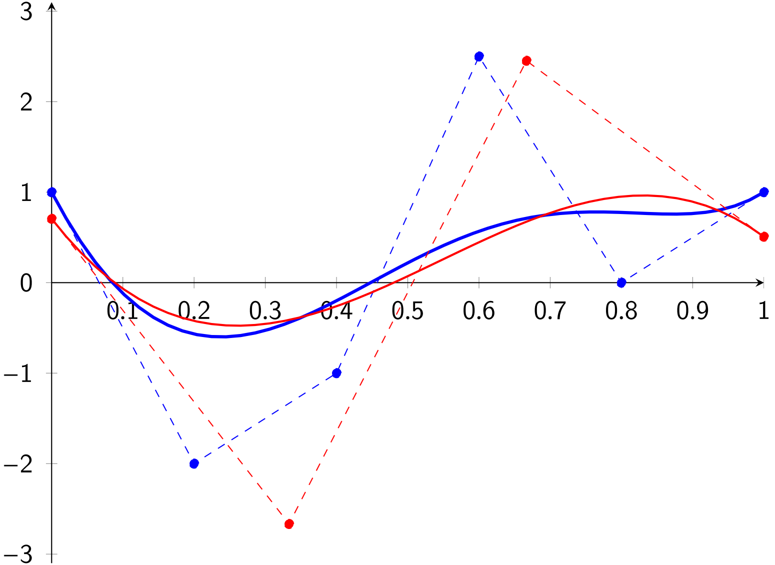

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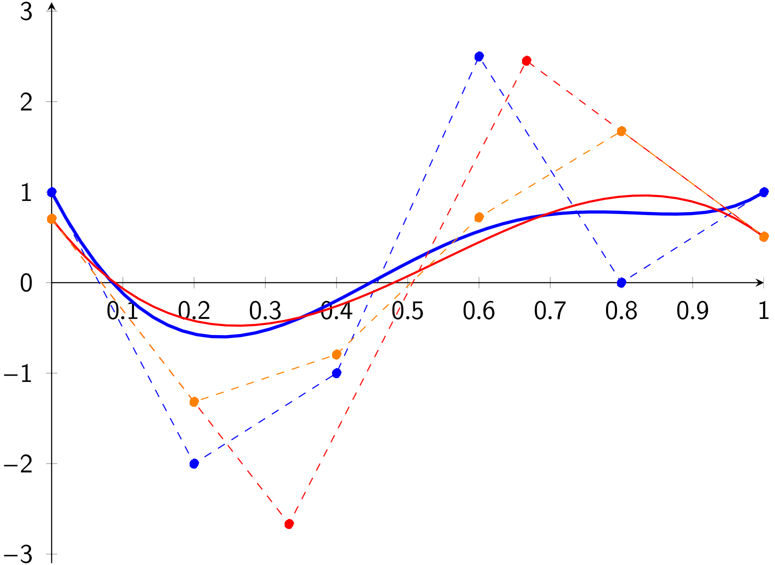

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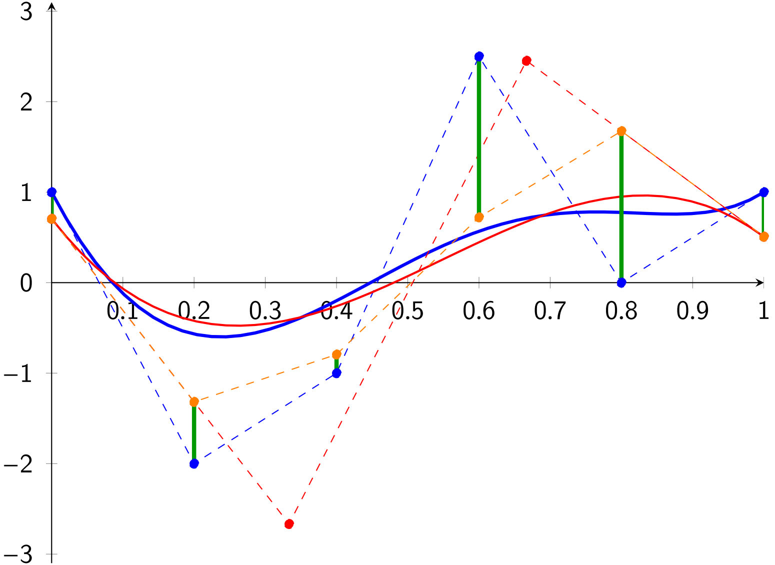

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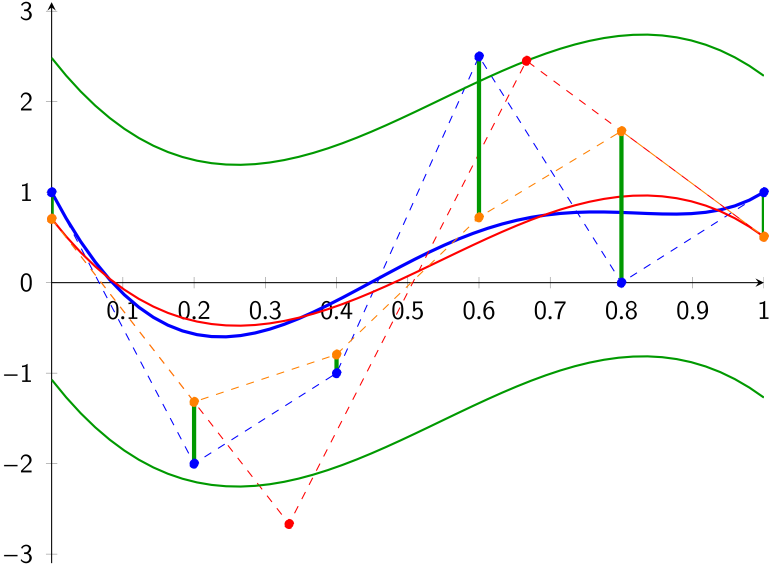

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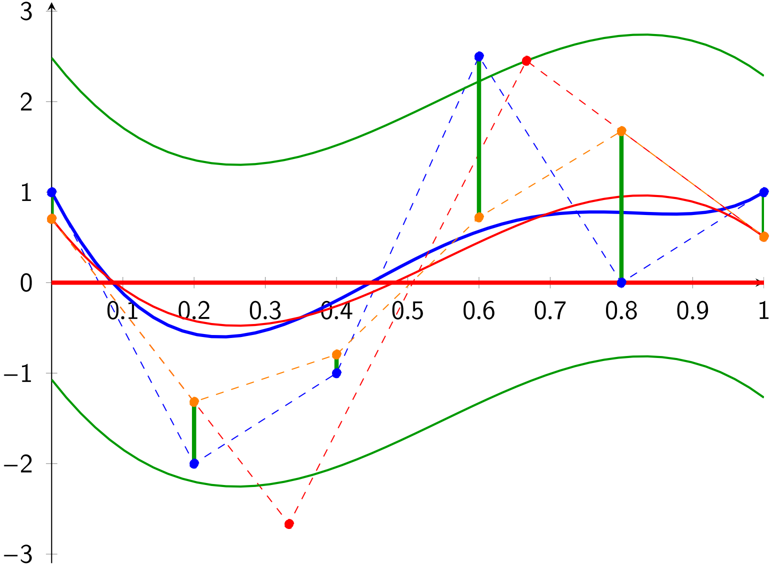
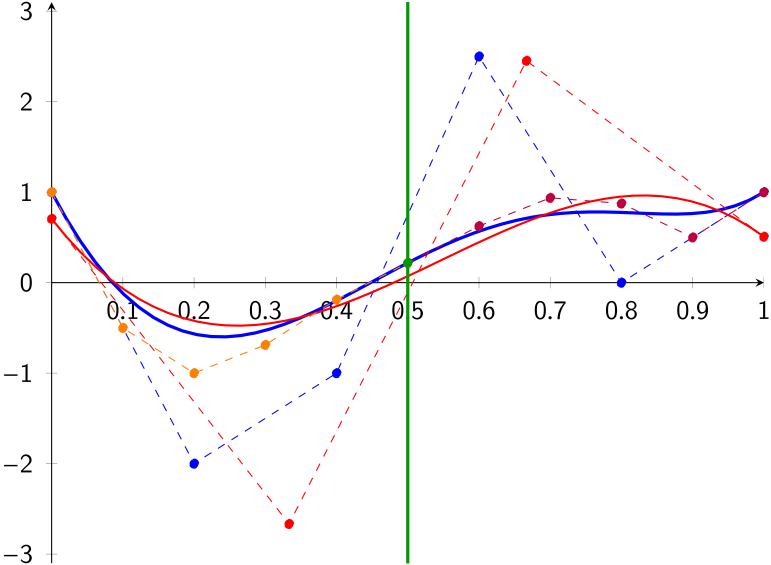
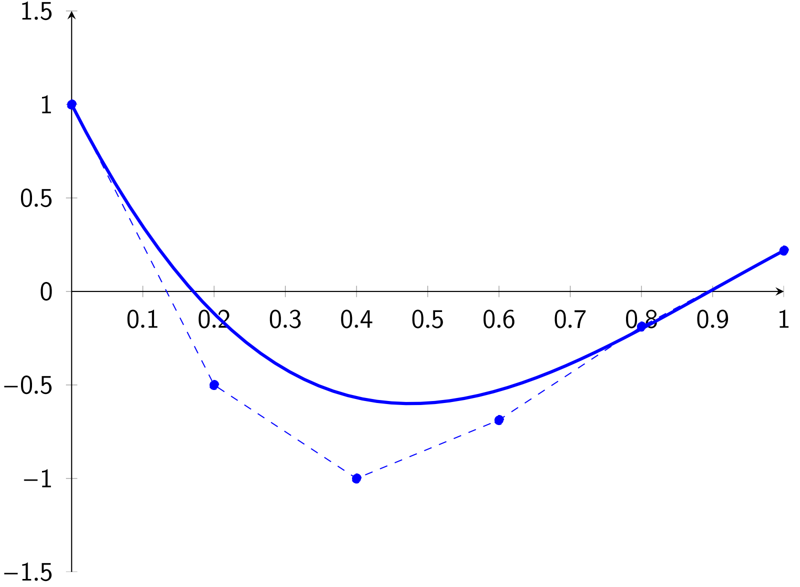


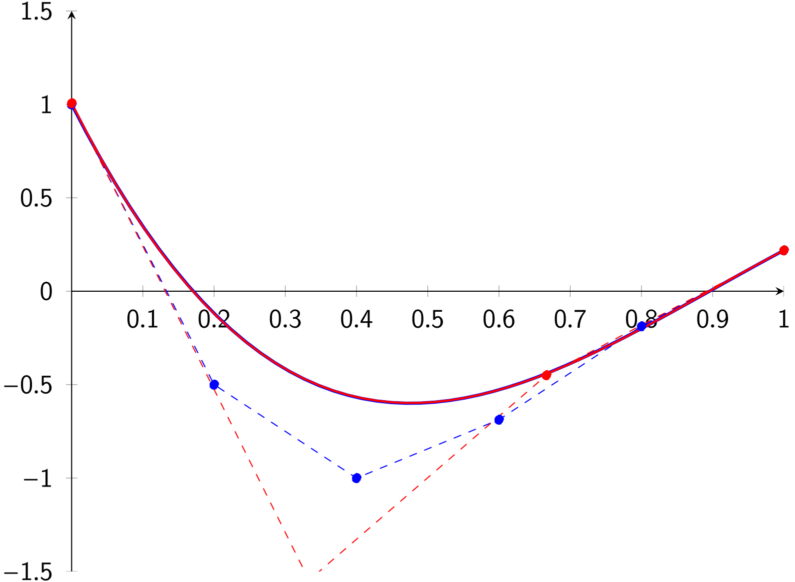
Der CubeClip Algorithmus



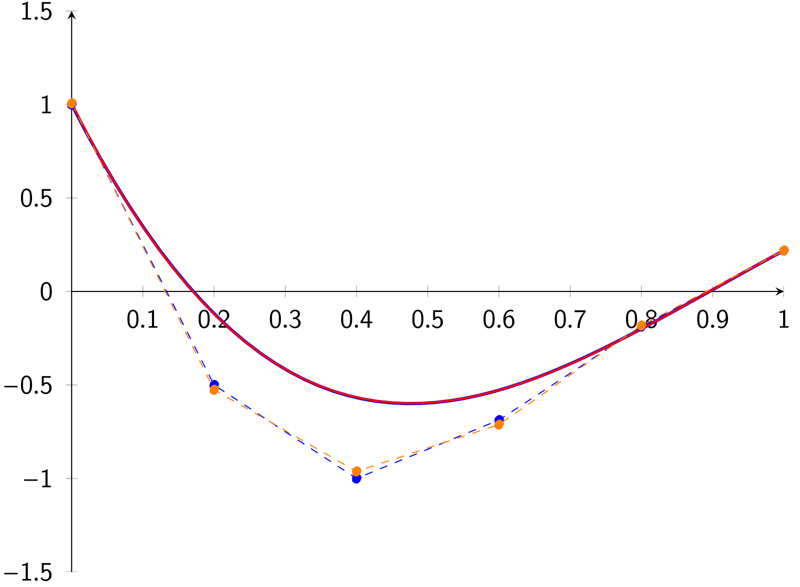
Der CubeClip Algorithmus



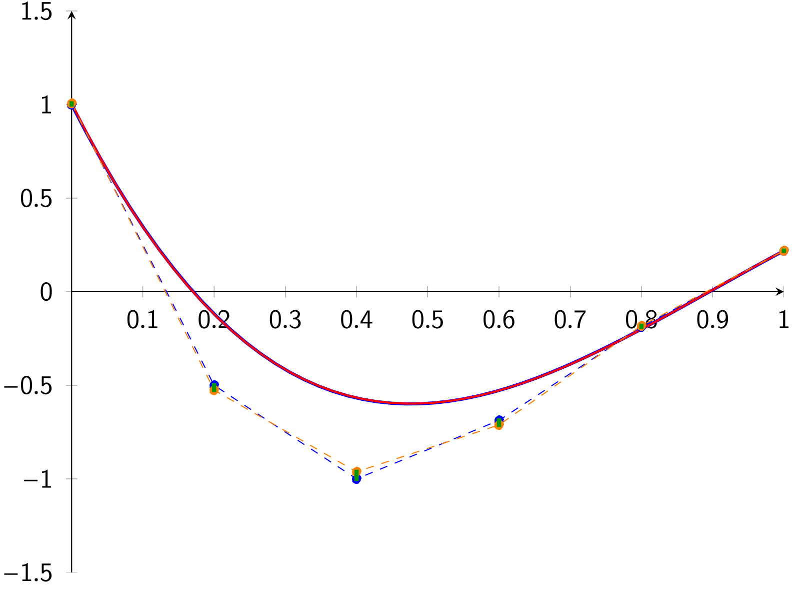
Der CubeClip Algorithmus



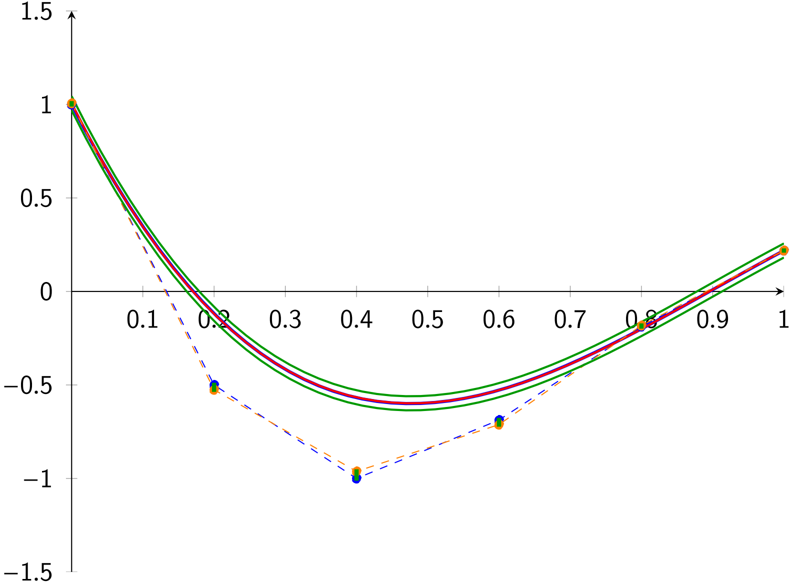
Der CubeClip Algorithmus



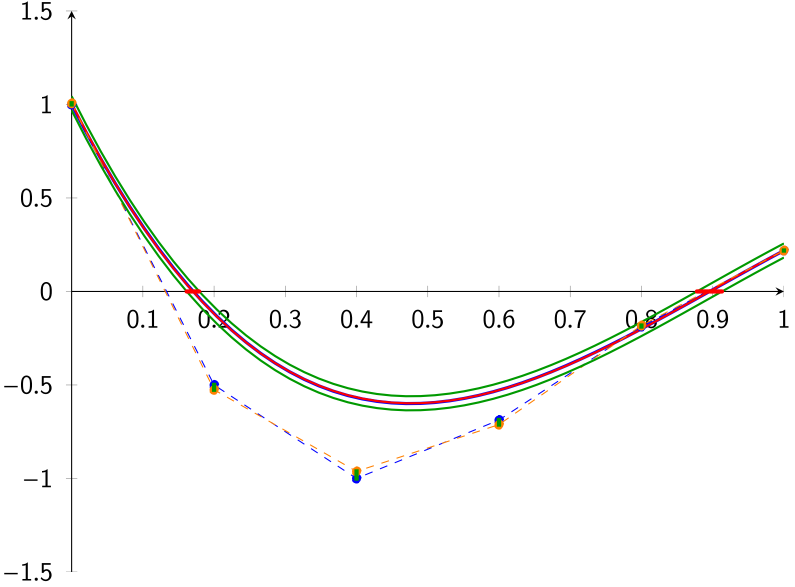
Der CubeClip Algorithmus



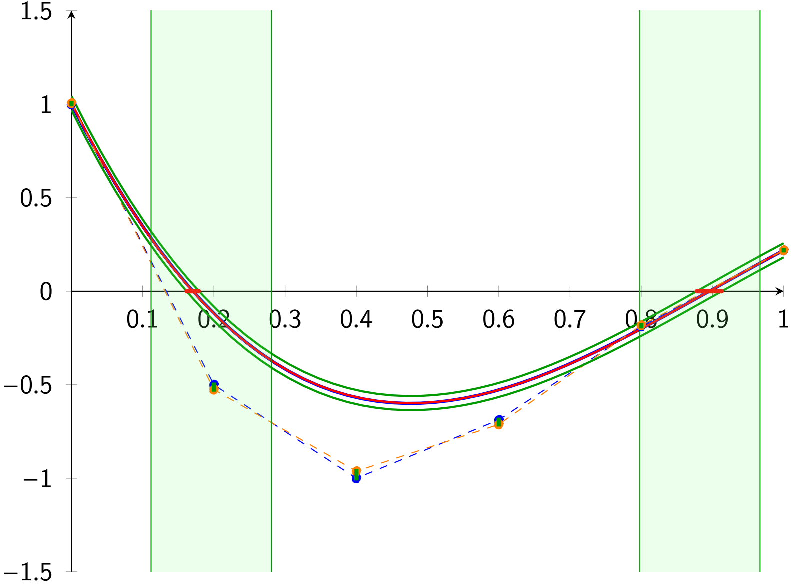
Der CubeClip Algorithmus



Der CubeClip Algorithmus



Der CubeClip Algorithmus



Übersicht

Untersuchte Polynome

$$f_2 := (t - \tfrac{1}{3})(3 - t)$$
$$f_4 := (t - \tfrac{1}{3})(2 - t)(t + 5)^2$$
$$f_8 := (t - \tfrac{1}{3})(2 - t)^3(t + 5)^4$$
$$f_{16} := (t - \tfrac{1}{3})(2 - t)^5(t + 5)^{10}$$

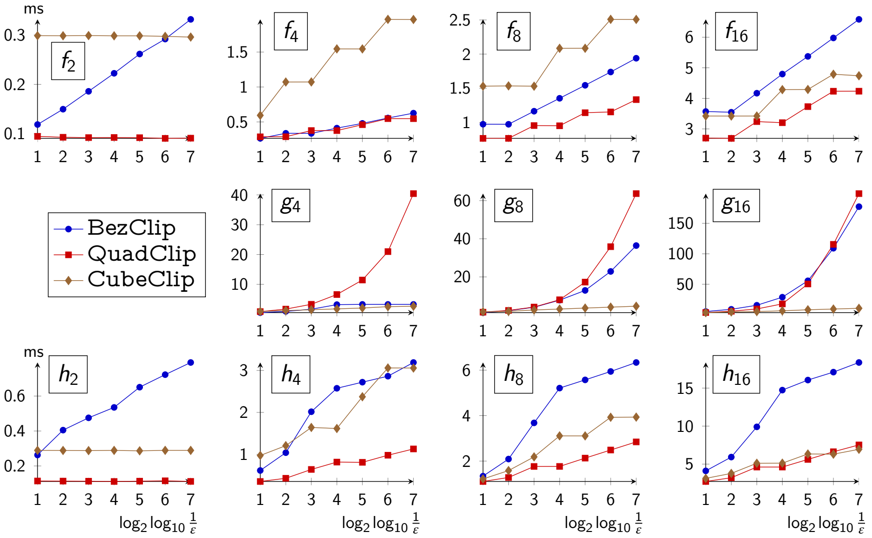
$$g_4 := (t - \tfrac{1}{3})^3(t - 5)$$
$$g_8 := (t - \tfrac{1}{3})^3(2 + t)^3(t - 5)^2$$
$$g_{16} := (t - \tfrac{1}{3})^3(2 + t)^2(t - 5)^7(t + 7)^4$$

$$h_2 := (t - 0.56)(t - 0.57)$$
$$h_4 := (t - 0.4)(t - 0.40000001)(t + 1)(2 - t)$$
$$h_8 := (t - 0.50000002)(t - 0.50000003)(t + 5)^3(t + 7)^3$$
$$h_{16} := (t - 0.30000008)(t - 0.30000009)(6 - t)^7(t + 5)^6(t + 7)$$

Rekursionstiefe

ε	10^{-2}			10^{-4}			10^{-8}			10^{-16}			10^{-32}			10^{-64}			10^{-128}		
Alg	B	Q	C	B	Q	C	B	Q	C	B	Q	C	B	Q	C	B	Q	C	B	Q	C
f_2	2	1	1	3	1	1	4	1	1	5	1	1	6	1	1	7	1	1	8	1	1
f_4	2	2	1	3	2	2	3	3	2	4	3	3	5	4	3	6	5	4	7	5	4
f_8	3	2	2	3	2	2	4	3	2	5	3	3	6	4	3	7	4	4	8	5	4
f_{16}	3	2	2	3	2	2	4	3	2	5	3	3	6	4	3	7	5	4	8	5	4
g_4	7	7	5	14	14	7	27	27	9	54	54	11	55	103	13	55	200	16	55	396	17
g_8	7	7	5	14	14	6	27	27	9	54	54	11	81	94	13	134	174	15	205	276	17
g_{16}	6	7	3	12	14	5	23	27	6	45	54	8	90	94	11	179	174	13	293	277	15
h_2	7	1	1	9	1	1	10	1	1	11	1	1	12	1	1	13	1	1	14	1	1
h_4	7	3	3	13	4	4	22	6	5	26	7	5	27	7	6	28	8	7	30	9	7
h_8	5	4	2	9	5	3	18	7	4	22	7	5	23	8	5	24	9	6	25	10	6
h_{16}	4	2	2	7	3	3	14	5	4	18	5	4	19	6	5	20	7	5	21	8	6

Rechenzeit



ms
f_2
f_4
f_8
f_{16}
g_4
g_8
g_{16}
h_2
h_4
h_8
h_{16}
BezClip
QuadClip
CubeClip
$\log_2 \log_{10} \frac{1}{\varepsilon}$

Ende

Danke für die Aufmerksamkeit.

Demo!

Appendix: Übersicht

Bestapproximationssatz

Satz aus Numerischer Mathematik 1:

Für $p \in \Pi_N$ ist $q = \sum_{i=0}^{n} \alpha_i B_i$ Bestapproximation an Π_n, wenn für alle $j = 0, \ldots, n$

$$0 = \langle p - q, B_j \rangle = \left\langle p - \sum_{i=0}^{n} \alpha_i B_i, B_j \right\rangle,$$

also wenn

$$\sum_{i=0}^{n} \alpha_i \langle B_i, B_j \rangle = \langle p, B_j \rangle$$

für alle $j = 0, \ldots, n$.

Lemma zum Beweis

$$\sum_{k=0}^{n} \langle f, B_k \rangle \langle D_k, D_j \rangle$$

Lemma zum Beweis

$$\sum_{k=0}^{n} \langle f, B_k \rangle \langle D_k, D_j \rangle$$
$$= \sum_{k=0}^{n} \langle f, B_k \rangle \left\langle \sum_{i=0}^{n} c_{k,i} B_i, D_j \right\rangle$$

Lemma zum Beweis

$$
\begin{aligned}
&\sum_{k=0}^{n} \langle f, B_k \rangle \langle D_k, D_j \rangle \\
&\qquad = \sum_{k=0}^{n} \langle f, B_k \rangle \left\langle \sum_{i=0}^{n} c_{k,i} B_i, D_j \right\rangle \\
&\qquad = \sum_{k=0}^{n} \langle f, B_k \rangle \sum_{i=0}^{n} c_{k,i} \langle B_i, D_j \rangle
\end{aligned}
$$

Lemma zum Beweis

$$
\begin{aligned}
&\sum_{k=0}^{n} \langle f, B_k \rangle \langle D_k, D_j \rangle \\
&\quad = \sum_{k=0}^{n} \langle f, B_k \rangle \left\langle \sum_{i=0}^{n} c_{k,i} B_i, D_j \right\rangle \\
&\quad = \sum_{k=0}^{n} \langle f, B_k \rangle \sum_{i=0}^{n} c_{k,i} \langle B_i, D_j \rangle \\
&\quad = \sum_{k=0}^{n} \langle f, B_k \rangle c_{k,j}
\end{aligned}
$$

Lemma zum Beweis

$$\sum_{k=0}^{n} \langle f, B_k \rangle \langle D_k, D_j \rangle$$

$$= \sum_{k=0}^{n} \langle f, B_k \rangle \left\langle \sum_{i=0}^{n} c_{k,i} B_i, D_j \right\rangle$$

$$= \sum_{k=0}^{n} \langle f, B_k \rangle \sum_{i=0}^{n} c_{k,i} \langle B_i, D_j \rangle$$

$$= \sum_{k=0}^{n} \langle f, B_k \rangle c_{k,j}$$

$$= \langle f, \sum_{k=0}^{n} c_{k,j} B_k \rangle$$

Lemma zum Beweis

$$\begin{aligned}
\sum_{k=0}^{n} \langle f, B_k\rangle \langle D_k, D_j\rangle \\
&= \sum_{k=0}^{n} \langle f, B_k\rangle \left\langle \sum_{i=0}^{n} c_{k,i} B_i, D_j \right\rangle \\
&= \sum_{k=0}^{n} \langle f, B_k\rangle \sum_{i=0}^{n} c_{k,i} \langle B_i, D_j\rangle \\
&= \sum_{k=0}^{n} \langle f, B_k\rangle c_{k,j} \\
&= \langle f, \sum_{k=0}^{n} c_{k,j} B_k\rangle
\end{aligned}$$

Lemma zum Beweis

$$\begin{aligned}
&\sum_{k=0}^{n} \langle f, B_k \rangle \langle D_k, D_j \rangle \\
&\quad = \sum_{k=0}^{n} \langle f, B_k \rangle \left\langle \sum_{i=0}^{n} c_{k,i} B_i, D_j \right\rangle \\
&\quad = \sum_{k=0}^{n} \langle f, B_k \rangle \sum_{i=0}^{n} c_{k,i} \langle B_i, D_j \rangle \\
&\quad = \sum_{k=0}^{n} \langle f, B_k \rangle c_{k,j} \\
&\quad = \langle f, \sum_{k=0}^{n} c_{j,k} B_k \rangle
\end{aligned}$$

Lemma zum Beweis

$$\begin{aligned}
\sum_{k=0}^{n} \langle f, B_k \rangle \langle D_k, D_j \rangle
&= \sum_{k=0}^{n} \langle f, B_k \rangle \left\langle \sum_{i=0}^{n} c_{k,i} B_i, D_j \right\rangle \\
&= \sum_{k=0}^{n} \langle f, B_k \rangle \sum_{i=0}^{n} c_{k,i} \langle B_i, D_j \rangle \\
&= \sum_{k=0}^{n} \langle f, B_k \rangle c_{k,j} \\
&= \langle f, \sum_{k=0}^{n} c_{j,k} B_k \rangle = \langle f, D_j \rangle
\end{aligned}$$

Lemma zum Beweis

$$\begin{aligned}
&\sum_{k=0}^{n} \langle f, B_k\rangle \langle D_k, D_j\rangle \\
&\quad = \sum_{k=0}^{n} \langle f, B_k\rangle \left\langle \sum_{i=0}^{n} c_{k,i} B_i, D_j \right\rangle \\
&\quad = \sum_{k=0}^{n} \langle f, B_k\rangle \sum_{i=0}^{n} c_{k,i} \langle B_i, D_j\rangle \\
&\quad = \sum_{k=0}^{n} \langle f, B_k\rangle c_{k,j} \\
&\quad = \langle f, \sum_{k=0}^{n} c_{j,k} B_k\rangle = \langle f, D_j\rangle
\end{aligned}$$

Direkte Formel für $q(t) = \sum_{j=0}^{n} \alpha_j B_{j,n}(t)$

Wir haben für $p = \sum_{i=0}^{N} b_i B_{i,N}$:

$$\alpha_j = \langle p, D_{j,n} \rangle = \left\langle \sum_{i=0}^{N} b_i B_{i,N}, D_{j,n} \right\rangle$$

$$= \sum_{i=0}^{N} b_i \left\langle B_{i,N}, D_{j,n} \right\rangle$$

Direkte Formel für $q(t) = \sum_{j=0}^{n} \alpha_j B_{j,n}(t)$

Wir haben für $p = \sum_{i=0}^{N} b_i B_{i,N}$:

$$\alpha_j = \langle p, D_{j,n} \rangle$$

$$= \sum_{i=0}^{N} b_i \left\langle B_{i,N}, D_{j,n} \right\rangle = \sum_{i=0}^{N} b_i \beta_{i,j}^{(N,n)}$$

Direkte Formel für $q(t) = \sum_{j=0}^{n} \alpha_j B_{j,n}(t)$

Wir haben für $p = \sum_{i=0}^{N} b_i B_{i,N}$:

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$$= \sum_{i=0}^{N} b_i \left\langle B_{i,N}, D_{j,n} \right\rangle = \sum_{i=0}^{N} b_i \beta_{i,j}^{(N,n)}$$

$$\text{mit } M^{(N,n)} := (\beta_{i,j}^{(N,n)}) = \left(\left\langle B_{i,N}, D_{j,n} \right\rangle\right)_{\substack{i=0,\ldots,N \\ j=0,\ldots,n}}.$$

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Wir haben für $p = \sum_{i=0}^{N} b_i B_{i,N}$:

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$$\text{mit } M^{(N,n)} := (\beta_{i,j}^{(N,n)}) = (\left\langle B_{i,N}, D_{j,n} \right\rangle)_{\substack{i=0,\dots,N \\ j=0,\dots,n}}.$$

$$(M^{(N,n)})^{\mathrm{t}} \begin{pmatrix} b_0 \\ \vdots \\ b_N \end{pmatrix} = \begin{pmatrix} \beta_{0,0}^{(N,n)} & \cdots & \beta_{N,0}^{(N,n)} \\ \vdots & \ddots & \vdots \\ \beta_{0,n}^{(N,n)} & \cdots & \beta_{N,n}^{(N,n)} \end{pmatrix} \begin{pmatrix} b_0 \\ \vdots \\ b_N \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_n \end{pmatrix}$$